4. $\sum a_n$, $\sum b_n$ positive
$\sum b_n$ diverges

a) $a_n > b_n$ then $\sum a_n$ diverges

b) $a_n < b_n$ then $\sum a_n$ unknown

6. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}$

$$\int_1^{\infty} \frac{1}{\sqrt[4]{x}}\,dx = \int_1^{\infty} x^{-1/4}\,dx = \frac{4}{3}x^{3/4}\Big|_1^{\infty} = \infty$$

diverges

7. $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges

$$\int_1^{\infty} \frac{1}{x^4}\,dx = \int_1^{\infty} x^{-4}\,dx = -\frac{1}{3}x^{-3}\Big|_1^{\infty} = +\frac{1}{3}$$

8. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \arctan(x)\Big|_1^{\infty} = \frac{\pi}{2} - \tan^{-1}(1)$$

9. $\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$ converges

$$\int_1^{\infty} \frac{1}{x^2+x+1} = \int_1^{\infty} \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}$$

$$= \frac{4}{3}\arctan\left(\frac{x+\frac{1}{2}}{\frac{3}{4}}\right)\Big|_1^{\infty} < \infty$$

10. $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$ diverges

$$\int_2^{\infty} \frac{\sqrt{x}}{x-1} = \int \frac{\sqrt{x}}{(\sqrt{x}-1)(\sqrt{x}+1)}\,dx \qquad u=\sqrt{x},\ du = \frac{1}{2\sqrt{x}}\,dx$$

$$= \int \frac{2u^2}{(u-1)(u+1)}\,du = \int 2 + \frac{2}{(u-1)(u+1)}\,du$$

$$= \int 2 + \frac{-1}{u+1} + \frac{1}{u-1}\,du$$

$$= 2u - \ln(u+1) + \ln(u-1)$$

$$= 2\sqrt{x} - \ln(\sqrt{x}+1) + \ln(\sqrt{x}-1)\Big|_2^{\infty}$$

$$= \underbrace{2\sqrt{x}}_{\to\infty} + \underbrace{\ln\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)}_{\to 1}\Big|_2^{\infty} = \infty$$

11. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \ldots$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^3 = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

converges

14. $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ compare to $\sum \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\left(\frac{n^2}{n^3+1}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3}{n^3+1} = 1$$

$\sum \frac{1}{n}$ diverges so $\sum \frac{n^2}{n^3+1}$ diverges

19. $\sum_{n=1}^{\infty} \frac{n-1}{n4^n}$

$$\frac{n-1}{n4^n} = \frac{n}{n4^n} - \frac{1}{n4^n} = \frac{1}{4^n} - \frac{1}{n4^n}$$

$\sum \left(\frac{1}{4}\right)^n$ geometric converges $\left|\frac{1}{4}\right| < 1$

$\sum \frac{1}{n4^n} \le \sum \frac{1}{4^n}$ converges

So $\sum \frac{n-1}{n4^n} = \sum \left(\frac{1}{4}\right)^n - \sum \frac{1}{n4^n}$ converges

30. Find $\sum_{n=1}^{\infty} \frac{1}{n^5}$ 3 decimal places

$$S_5 = \sum_{n=1}^{5} \frac{1}{n^5} = 1.0366617888$$

$$S_{10} = \sum_{n=1}^{10} \frac{1}{n^5} = 1.03690734134$$

truncated: 1.036

rounded: 1.037